Homoclinic orbits in the near-integrable double discrete sine-Gordon equation

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Abstract
We establish the existence of homoclinic orbits for the near-integrable double discrete sine-Gordon (dDSG) equation under periodic boundary conditions. The hyperbolic structure and homoclinic orbits are constructed through the Bäcklund transformation and Lax pair. A geometric perturbation method based on Mel'nikov analysis is used to establish necessary criteria for the persistent of temporally homoclinic orbits for the class of dDSG equations with dissipative perturbations.

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1. Introduction

In recent years there has been remarkable progress on discrete integrable systems and associated difference equations. This paper is concerned with the existence of homoclinic orbits of the near-integrable double discrete sine-Gordon (dDSG) equation. A starting point is an *integrable lattice* version of the SG equation, which is a nonlinear partial difference equation (PΔE), i.e. a system in which both the spatial—as well as the time variable is discrete. Discrete spacetime model in soliton theory have acquired a prominent role in mathematical physics. The first examples of such models were proposed by Hirota [10] as discrete analogues of the major continuum soliton models. Subsequent development was carried out mostly by a Dutch group (see Nijhoff and Capel [18] and references therein). The lattice sine-Gordon equation (dDSG) that we want to investigate in this paper was first presented by Hirota [10] and therefore is sometimes referred to as the Hirota equation. The particular discretization scheme of Hirota is used by Orfanidis [19] to study the discrete nonlinear σ-model from a group-theoretical viewpoint. We show that the dDSG equation with periodic boundary conditions in the discrete space admits solutions which are homoclinic in the 'discrete' time to the hyperbolic fixed point. We consider here the dDSG as a discrete dynamical system—a mapping with two indices n and m. For a specialist in dynamical systems, the Hirota equation gives a family of examples of

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symplectic maps of finite-dimensional phase space—the so-called standard map [3, 11]. Also, the dDSG plays an important role in the discrete geometry. The continuum limit describes surfaces with constant negative Gaussian curvature. The dDSG (classical) model possesses a nice geometric interpretation which parallels the continuum theory to a surprising extent [4], and which provides a useful guiding principle for the analysis of the model from a geometrical viewpoint.

We present necessary criteria for the persistence of homoclinic structure of the dDSG equation with small dissipative perturbations, tending to the continuous perturbed SG equation for which we have established the persistence of homoclinic orbits and its temporal chaotic behaviour (see [21] for a geometric approach, [16] for an analytic approach and [15] for a numerical investigation). With the method being developed here we exploit two facts: *first* we investigate the existence of homoclinic orbits associated with a hyperbolic fixed point of the unperturbed dDSG through a Bäcklund transformation and *second* we establish the persistence of these orbits under small dissipative perturbations. Our results are therefore interested for the discrete K-surface with small perturbations (numerical simulations of discrete geometry), related to the new directions of the combination lattice equations and non-integrability criteria. We note that the dynamics of the dDSG equation is very close to the continuum model. If the underlying dimension of some system under consideration is infinite (for example the SG model), we are faced with two difficulties: *first* the techniques are largely unavailable for performing the qualitative analysis and *second* the phenomena occurring can be considerably more complicated. To resolve this apparent intractability of higher-dimensional problems, techniques involving reductions to a lower-dimensional space are very attractive. Recently, Haller has been developing an alternative perspective, based on the existence of multi-pulse homoclinic solutions near resonance, which has been applied to finite-dimensional *spatial* discretizations of the perturbed nonlinear Schrödinger (NLS) equations and to the partial differential equations (PDEs) (see [9] and references therein).

This paper is organized in the following manner. In section 2, we formulate the problem. In section 3, we review some basic concepts of the dDSG as a nonlinear PΔE and its relationship with discrete K-surfaces, we present the Lax pair of dDSG and find an analytic expression of the homoclinic orbits through the Bäcklund transformation. We obtain an explicit formula for the gradient of the individual constants of motion evaluated on the homoclinic orbits. In section 4, we derive necessary conditions for the persistence of homoclinic solutions of the dDSG equation under dissipative perturbations based on the Mel'nikov analysis and geometric arguments of our problem.

2. Formulation of the problem

The sine-Gordon (SG) equation

$$u_{xx} - u_{tt} = \sin u \qquad\qquad (2.1)$$

is one of the simplest integrable PDEs with many physical and mathematical applications and admits solutions for very special initial data and 2π-periodic boundary conditions in space which are homoclinic in time. These homoclinic structures persist under dissipative perturbations and the near-integrable equation possesses temporal chaotic behaviour [21].

When we integrate the SG equations numerically, one considers the spatial variable to be a lattice of points, so the objects of study are sets on nonlinear ordinary differential equations with time remaining a continuous independent variable. Such problems have a venerable place in the study of nonlinear waves going back to the Frenkel–Kontorova model (discrete in *space* version of the sine-Gordon equation), the Fermi–Pasta–Ulam model. In recent studies of the

nonlinear dynamics of lattice systems consisting of coupled oscillators attention was paid to time-periodic and spatially localized excitations—discrete breathers—a prototype of such a nonlinear lattice is represented by the discrete NLS equations. Dealing with the discrete lattice structure and thus with a network of coupled oscillators one faces the fact that most nonlinear lattice systems are non-integrable (see [14] and references therein).

One of the aspects that make integrable equations very special is that are can find discrete analogues of these equations, sine-Gordon, KdV for instance, tending to the original continuous equations in special limits and that are integrable themselves. These discrete analogues can be either ordinary or partial difference equations such as the MacMillan map [8] or mixed differential-difference equations such as the Ablowitz–Ladik [1] and the Toda lattice [23].

We consider the *integrable* discrete analogue of equation (2.1):

$$\left(1 - \tfrac{1}{4}h^2\right) \tan \tfrac{1}{4}\left(\theta_{m+1}^n + \theta_{m-1}^n\right) = \left(1 + \tfrac{1}{4}h^2\right) \tan \tfrac{1}{4}\left(\theta_m^{n+1} + \theta_m^{n-1}\right) \tag{2.2}$$

as a discrete dynamical system and we establish the persistence of its homoclinic structure under small dissipative perturbations.

Equation (2.2) (first appeared in Hirota [10]) is a complete integrable discretization of the sine-Gordon equation (2.1), where $\theta_m^n := u(mh, nh)$ denotes the approximation of $u(x, t)$. We rewrite (2.2) in the form

$$\theta_m^{n+1} + \theta_m^{n-1} = \theta_{m+1}^n + \theta_{m-1}^n + 2i \ln \left\{ \frac{1 + \tfrac{1}{4}h^2 \exp\left[\tfrac{1}{2}i(\theta_{m+1}^n + \theta_{m-1}^n)\right]}{1 + \tfrac{1}{4}h^2 \exp\left[-\tfrac{1}{2}i(\theta_{m+1}^n + \theta_{m-1}^n)\right]} \right\} \tag{2.3}$$

or using the identity $\mathrm{Arg}(z) = \vartheta = \tfrac{1}{2i}\ln(z\bar{z}^{-1})$, $z = re^{i\vartheta}$, we rewrite equation (2.3) as follows:

$$\theta_m^{n+1} + \theta_m^{n-1} = \theta_{m+1}^n + \theta_{m-1}^n + 2i \, \mathrm{Arg}\left\{1 + \tfrac{1}{4}h^2 \exp\left[\tfrac{1}{2}i(\theta_{m+1}^n + \theta_{m-1}^n)\right]\right\}. \tag{2.4}$$

We consider the above equations (2.3) or (2.4) as a discrete evolution equation with respect to index n (in the continuum limit approaches the time variable t):

$$\theta_m^{n+1} = \boldsymbol{F}(\theta_m^{n-1}, \theta_{m+1}^n, \theta_{m-1}^n) \tag{2.5}$$

and we assume that θ_m^n satisfies the periodic and even boundary conditions

$$\theta_{m+M}^n = \theta_m^n \qquad \theta_{M-m}^n = \theta_m^n \qquad M = 2p \quad n \in \mathbb{Z} \quad m = 0, 1, \ldots, M - 1 \quad h = \frac{2}{\pi}. \tag{2.6}$$

Faddeev and Volkov [7] studied the symplectic structure of Hirota's equation on a spatially periodic lattice of even length $M = 2p$. We observe from (2.3) that θ_m^{n+1} only depends on $\theta_{m-1}^n, \theta_{m+1}^n$ and θ_m^{n-1}, there is no contribution from θ_m^n. Assuming $M = 2p$, we specify initial values $\{\theta_m^0, m = 1, 3, \ldots, 2p - 1\}$ and $\{\theta_m^n, m = 2, 4, \ldots, 2p\}$ and these values completely determine the solution for all $n \in \mathbb{Z}$. However, we may also consider the following initial values $\{\theta_m^0, m = 2, 4, \ldots, 2p\}$ and $\{\theta_m^1, m = 1, 3, \ldots, 2p - 1\}$ which together with (2.3) again completely determine the solution for all times.

The phase space $\mathcal{L}$ of dDSG (2.3) is $\mathbb{Z}^2$ with the interpretation that edge points in lightcone directions. If we start with space-periodic initial data θ_j^i along a saw C_n on the lightcone lattice (see figure 1) then the local evolution given by (2.3) (or (2.4)) will determine the function $\theta : \mathcal{L} \to \mathbb{R}$ on the whole lattice.

In the smooth case the SG model describes the angle between the asymptotic line on an asymptotically parametrized K-surface. Bobenko and Pinkal [4] studied the geometric properties of the *discrete K*-surfaces (maps $\mathbb{Z}^2 \to \mathbb{R}^3$) and proved that these surfaces are parametrized by the second-order difference equation

$$\theta_m^{n+1} + \theta_m^{n-1} = \theta_{m+1}^n + \theta_{m-1}^n + 2i \, \mathrm{Arg}\left\{1 + \tfrac{1}{4}h^2 \exp\left[i\theta_{m+1}^n\right]\right\} + +2i \, \mathrm{Arg}\left\{1 + \tfrac{1}{4}h^2 \exp\left[i\theta_{m-1}^n\right]\right\} \tag{2.7}$$
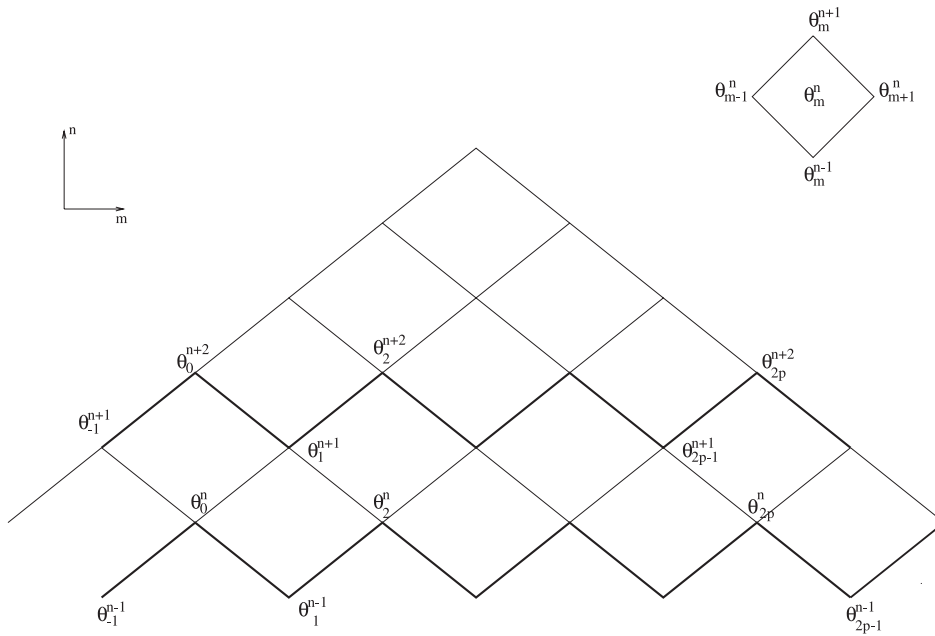
Figure 1. The geometry of the double discrete SG equation.

which is equivalent to the Hirota equation (2.3).

In this paper, we consider the perturbed dDSG equation:

$$\theta_m^{n+1} + \theta_m^{n-1} = \theta_{m+1}^n + \theta_{m-1}^n + 2i \ln \left\{ \frac{1 + \frac{1}{4}h^2 \exp\left[\frac{1}{2}i(\theta_{m+1}^n + \theta_{m-1}^n)\right]}{1 + \frac{1}{4}h^2 \exp\left[\frac{1}{2}-i(\theta_{m+1}^n + \theta_{m-1}^n)\right]} \right\}$$

$$+ \varepsilon \left\{ b\,\theta_m^n + c\,\frac{\theta_{m+1}^n - 2\theta_m^n + \theta_{m-1}^n}{h^2} \right\} \tag{2.8}$$

where θ_m^n satisfy the periodic boundary conditions (2.6) and $0 < \varepsilon \ll 1, b \in (0, b_0), c \in (0, c_0), h = \frac{2\pi}{M} < 1$. When $\varepsilon = 0$, the unperturbed dDSG is a completely integrable lattice equation. It has a spatially uniform solution $\hat{\theta}_m^n = \pi$.

We study the homoclinic structure for the dDSG equation and derive an analytic expression of the homoclinic orbits through the Bäcklund transformation and its Lax pair formulation. The homoclinic orbits approach the hyperbolic fixed point $\theta_m^n = \pi$ as $n \to \pm\infty$.

The system (2.8) is the (classical) double discrete model of the following near-integrable SG equation:

$$u_{tt} - u_{xx} = \sin u + \varepsilon(bu + cu_{xx}) \tag{2.9}$$

when the lattice spacing h tends to zero and the variables θ turn to the smooth function $u(x, t)$, with $u(x + 2\pi, t) = u(x, t) = u(-x, t)$ and $\varepsilon \in (0, \varepsilon_1), b, c > 0$ (cf [2, 15, 21]). Thus, system (2.8) is of interest both as a perturbation of the completely *integrable lattice* equation (finite dimension), and as an approximation to the PDE (2.9).

We shall prove the persistence of homoclinic orbits for the system (2.8). The Mel'nikov method [12, 13, 17] is one of the techniques that has proved to be very useful in dynamical systems. The method establishes a way to measure the distance between the stable and unstable manifolds of a saddle-type invariant set for a perturbed integrable system that originally has homoclinic structure.

3. Integrability and homoclinic orbits

In this section, we present two results concerned with homoclinic structure of the dDSG equation. First, we construct the analytic expression of homoclinic orbits based on Bäcklund transformations of special solutions of the dDSG. Then, we obtain an explicit formula for the gradient of an important invariant of motion which we will use to build the distance function between the invariant manifolds.

3.1. Integrable background

The dDSG equation that we want to study is the following nonlinear PΔE:

$$\sin \frac{1}{4}\left(\theta_m^n + \theta_{m+1}^{n+1} + \theta_m^{n+1} + \theta_{m+1}^n\right) = \frac{p}{q} \sin \frac{1}{4}\left(\theta_m^n + \theta_{m+1}^{n+1} - \theta_m^{n+1} - \theta_{m+1}^n\right) \qquad (3.1)$$

with boundary conditions (2.6). We note that θ_m^n is the dynamical variable at site (n, m), $n, m \in \mathbb{Z}$, and $p, q \in \mathbb{C}$ are lattice parameters. Equation (3.1) is an alternative form of (2.2) or (2.3).

First, we note that the dDSG (3.1) arises from a discrete action principle. The action for the dDSG reads [5]

$$S = \sum_{n \in \mathbb{Z}} \sum_{m=0}^{M-1} \left\{ \theta_{m+1}^n (\theta_{m+1}^{n+1} + \theta_m^n) - \tfrac{1}{4} F\left(2\mathrm{i}(-1)^m (\theta_m^n + \theta_{m+1}^{n+1}) + \sigma_2\right) \right.$$

$$\left. + \tfrac{1}{4} F\left(2\mathrm{i}(-1)^m (\theta_m^n + \theta_{m+1}^{n+1}) + \sigma_1\right) \right\}$$

$$\equiv \sum_{n \in \mathbb{Z}} \sum_{m=0}^{M-1} L(\theta_m^n, \theta_{m+1}^n, \theta_{m+1}^{n+1}, \theta_m^{n+1}) \qquad (3.2)$$

in which the function F is related to the dilogarithm function

$$F(x) = \int_{-\infty}^{x} \ln(1 + \mathrm{e}^{\mathrm{i}\xi}) \, \mathrm{d}\xi$$

and where σ_1 and σ_2 are parameters related to the lattice parameters p, q:

$$\mathrm{e}^{\sigma_1} = \frac{q}{p} \qquad \mathrm{e}^{\sigma_2} = \frac{p}{q}.$$

The Euler–Lagrange equation for (3.2) which is obtained by variation of S with respect to the variables θ_m^n, i.e.

$$\frac{\delta S}{\delta \theta_m^n} = 0$$

leads to the equation

$$\theta_m^{n+1} + \theta_{m+1}^{n-1} + \theta_{m+1}^n + \theta_{m-1}^n + \frac{\mathrm{i}(-1)^m}{2} \ln \left\{ \frac{1 + \exp\left[\sigma_1 + 2\mathrm{i}(-1)^m (\theta_m^n + \theta_{m+1}^{n+1})\right]}{1 + \exp\left[\sigma_2 + 2\mathrm{i}(-1)^m (\theta_m^n + \theta_{m+1}^{n+1})\right]} \right\}$$

$$+ \frac{\mathrm{i}(-1)^m}{2} \ln \left\{ \frac{1 + \exp\left[\sigma_2 + 2\mathrm{i}(-1)^m (\theta_m^n + \theta_{m-1}^{n-1})\right]}{1 + \exp\left[\sigma_1 + 2\mathrm{i}(-1)^m (\theta_m^n + \theta_{m-1}^{n-1})\right]} \right\} = 0 \qquad (3.3)$$

which is a consequence of equation (3.1).

Another important aspect of the dDSG, directly related to the integrability of this system, arises as the compatibility condition of an overdetermined linear system, Lax pair [19, 20]:

$$\phi_{m+1}^n(k) = V_{n,m}(k)\,\phi_m^n \tag{3.4a}$$

$$\phi_m^{n+1}(k) = U_{n,m}(k)\,\phi_m^n \tag{3.4b}$$

where k denotes the spectral parameter and $V_{n,m}$ is given by

$$V_{n,m}(k) = \begin{pmatrix} \dfrac{q}{q-k} & \dfrac{v_{m+1}^n}{q-k} \\[2mm] \dfrac{k^2}{q-k}\dfrac{1}{v_m^n} & \dfrac{q}{q-k}\dfrac{v_{m+1}^n}{v_m^n} \end{pmatrix} \qquad v_m^n = \exp[2i\theta_m^n] \tag{3.5}$$

and where $U_{n,m}$ is given by a similar matrix obtained from (3.5) by making the replacement $q \to p$ and $(n, m+1) \to (n+1, m)$.

Let $M_{n,m}(k, \theta_m^n)$ be the fundamental matrix to the Lax pair, i.e. the matrix solution to (3.4a) and (3.4b) with the initial conditions M_0 being a 2×2 identity matrix. The characteristic polynomial

$$\det\left(M_{n,m}(k, \theta_m^n) - \lambda I\right) = 0 \tag{3.6}$$

defines an invariant algebraic curve. The substitution

$$\mathcal{M} = \lambda - \frac{\operatorname{trace} M_{n,m}(k)}{2}$$

reduces this curve to the hyperelliptic form [4]:

$$\mathcal{M}^2 = \left(\frac{\operatorname{trace} M_{n,m}(k)}{2}\right)^2 - \det M_{n,m}(k). \tag{3.7}$$

We define the function Δ as

$$\Delta : \mathbb{C} \times \mathcal{L} \longrightarrow \mathbb{C}$$
$$\Delta(k, \theta_m^n) = \operatorname{trace} M_{n,m}(M; k, \theta_m^n). \tag{3.8}$$

The spectral parameter $k \in \mathbb{C}$ fulfils the following condition:

$$-2D \leqslant \Delta(k, \theta_m^n) \leqslant 2D \tag{3.9}$$

where D is constant and a critical point k_c of Δ satisfies

$$\left.\frac{d\Delta}{dk}\right|_{(k_c, \theta_m^n)} = 0. \tag{3.10}$$

An important sequence of constant of motion $\mathcal{F}_j : \mathcal{L} \to \mathbb{C}$ is defined by

$$\mathcal{F}_j(\theta_m^n) = \frac{1}{D}\Delta(k_{c,j}; \theta_m^n).$$

3.2. Analytic expression of homoclinic orbits

We construct the homoclinic orbits for the dDSG through the Bäcklund transformation. For more complete studies on Bäcklund transformation see the book [22]. The unperturbed dDSG equation has an unstable hyperbolic fixed point at $\hat{\theta}_m^n = \pi$. We utilize the Bäcklund transformation for a concrete example, to generate homoclinic orbits through an iteration of the transformation to incorporate all the unstable and stable modes. The homoclinic orbits for the PDE SG have been constructed by Ercolani *et al* [6] through the Bäcklund transformation of fixed solutions.

Let ϕ_m^n denote a fix solution of the Lax pair at (θ_m^n, k_d), where k_d is the double critical point of the function Δ, i.e. satisfies the condition (3.10) and $\Delta(k_d; \theta_m^n) = \pm 2D$. We define the 2×2 transformation matrix $\Gamma_{n,m}$ by

$$\Gamma_{n,m} = \begin{pmatrix} -\bar{k} - \alpha_{n,m} & \beta_{n,m} \\ \gamma_{n,m} & -\bar{k} - \delta_{n,m} \end{pmatrix} \tag{3.11}$$

where

$$\alpha_{n,m} = \frac{k}{\mathbb{D}} \left(|\phi_m^{n,1}|^2 + \bar{k}^2 |\phi_m^{n,2}|^2 \right) \qquad \delta_{n,m} = \bar{\alpha}_{n,m}$$

$$\beta_{n,m} = \frac{\phi_m^{n,1} \bar{\phi}_m^{n,2}}{\mathbb{D}} (\bar{k}^2 - |k|^2) \qquad \gamma_{n,m} = -\bar{\beta}_{n,m} \qquad \mathbb{D} = |\phi_m^{n,1}|^2 + |k|^2 |\phi_m^{n,2}|^2.$$

We define

$$\mathcal{V}_m^n = -p \left(\bar{k} + \alpha_{n-1,m} + \left(1 - \frac{k^2}{p^2} \right) + \frac{p-k}{v_m^n} \beta_{n,m} \right) \left(\bar{\beta}_{n-1,m} - \frac{p-k}{p} \frac{\bar{k} + \bar{\alpha}_{n-1,m}}{v_m^{n-1}} \right)^{-1}. \tag{3.12}$$

Then

$$\Theta_m^n = \pi + \frac{1}{2i} \ln \mathcal{V}_m^n \qquad v_m^n = e^{2i\theta_m^n} \tag{3.13}$$

and

$$\Phi_m^n(k) = \Gamma_{n,m}(k_d) \, \phi_m^n(k). \tag{3.14}$$

Proposition 1. *Let θ_m^n denote a solution of the dDSG with the double critical point k_d. We denote ϕ_m^n the general solution of the Lax pair at (θ_m^n, k_d). We define Θ_m^n and $\Phi_m^n(k)$ by (3.13) and (3.14).*

(a) Θ_m^n is also a solution of dDSG,
(b) $\Phi_m^n(k)$ solves the Lax pair (3.4a), (3.4b) at (Θ_m^n, k),
(c) Θ_m^n is homoclinic to the fixed solution $\hat{\theta}_m^n = \pi$, such that $\lim\limits_{n \to \pm\infty} \Theta_m^n = \pi$.

Proof. We show that $\Phi_m^n(k)$ solves the Lax pair at (Θ_m^n, k) provided that ϕ_m^n solves (3.4a), (3.4b) at (θ_m^n, k). From the equations of Lax pair we obtain

$$\Phi_{m+1}^n - V_{n,m}(\Theta_m^n)\Phi_m^n = \left[\Gamma_{n,m+1} V_{n,m}(\theta_m^n) - V_{n,m}(\Theta_m^n)\Gamma_{n,m} \right]\Phi_m^n$$

$$\Phi_m^{n+1} - U_{n,m}(\Theta_m^n)\Phi_m^n = \left[\Gamma_{n+1,m} U_{n,m}(\theta_m^n) - U_{n,m}(\Theta_m^n)\Gamma_{n,m} \right]\Phi_m^n.$$

Hence if

$$\Gamma_{n,m+1} V_{n,m}(\theta_m^n) - V_{n,m}(\Theta_m^n)\Gamma_{n,m} = 0 \tag{3.15a}$$

$$\Gamma_{n+1,m} U_{n,m}(\theta_m^n) - U_{n,m}(\Theta_m^n)\Gamma_{n,m} = 0 \tag{3.15b}$$

the proposition is established. In the component form equations (3.15a) and (3.15b) together the definitions (3.11)–(3.13) are equivalent to the eight algebraic equations. Indeed, the system

$$\Gamma_{n,m+1} = V_{n,m}(\Theta_m^n)\Gamma_{n,m} V_{n,m}^{-1}(\theta_m^n)$$

$$\Gamma_{n+1,m} = U_{n,m}(\Theta_m^n)\Gamma_{n,m} U_{n,m}^{-1}(\theta_m^n)$$

yields the following set of algebraic equations:

$$\mathcal{P}(\bar{k} + \alpha_{n+1,m}) = \left(\bar{k} + \alpha_{n,m} + \frac{p-k}{p}\frac{1}{v_m^{n+1}}\beta_{n,m}\right) - \frac{\mathcal{V}_m^{n+1}}{p}\left(\gamma_{n,m} + (p-k)\frac{(\bar{k}+\delta_{n,m})}{pv_m^{n+1}}\right)$$

$$\mathcal{P}\beta_{n+1,m} = (\bar{k}+\alpha_{n,m})\frac{v_m^n}{p} + \beta_{n,m}\frac{v_m^n}{v_m^{n+1}} - \frac{\mathcal{V}_m^{n+1}}{p}\left(\bar{\beta}_{n,m}\frac{v_m^n}{p} + (\bar{k}+\bar{\alpha}_{n,m})\frac{v_m^n}{v_m^{n+1}}\right)$$

$$\mathcal{P}\gamma_{n+1,m} = -\frac{k^2}{p}\frac{1}{\mathcal{V}_m^n}\left(\bar{k}+\alpha_{n,m} + \frac{\beta_{n,m}}{v_m^{n+1}}\frac{p}{p-k}\right) + \frac{\mathcal{V}_m^{n+1}}{\mathcal{V}_m^n}\left(\gamma_{n,m} + \frac{\bar{k}+\delta_{n,m}}{v^{n,m}}\frac{p-k}{p}\right)$$

$$\mathcal{P}(\bar{k}+\delta_{n+1,m}) = -\frac{k^2}{p\mathcal{V}_m^n}\left((\bar{k}+\alpha_{n,m})\frac{v_m^n}{p} + \beta_{n,m}\frac{v_m^n}{v_m^{n+1}}\right) + \frac{\mathcal{V}_m^{n+1}}{\mathcal{V}_m^n}\left(\gamma_{n,m}\frac{v_m^n}{p} + (\bar{k}+\delta_{n,m})\frac{v_m^n}{v_m^{n+1}}\right)$$

(3.16)

$$\mathcal{Q}(\bar{k}+\alpha_{n,m+1}) = \left(\bar{k}+\alpha_{n,m} + \frac{q-k}{q}\frac{\beta_{n,m+1}}{v_m^{n+1}}\right) - \frac{\mathcal{V}_{m+1}^n}{q}\left(\gamma_{n,m} + \frac{q-k}{q}\frac{\bar{k}+\delta_{n,m}}{v_{m+1}^n}\right)$$

$$\mathcal{Q}\beta_{n,m+1} = \left((\bar{k}+\alpha_{n,m})\frac{v_m^n}{q} + \beta_{n,m}\frac{v_m^n}{v_{m+1}^n}\right) - \frac{\mathcal{V}_{m+1}^n}{q}\left(\gamma_{n,m}\frac{v_m^n}{q} + (\bar{k}+\delta_{n,m})\frac{v_m^n}{v_{m+1}^n}\right)$$

$$\mathcal{Q}\gamma_{n,m+1} = -\frac{k^2}{q\mathcal{V}_m^n}\left(\bar{k}+\alpha_{n,m} + \frac{q-k}{q}\frac{\beta_{n,m}}{v_{m+1}^n}\right) + \frac{\mathcal{V}_{m+1}^n}{\mathcal{V}_m^n}\left(\gamma_{n,m} + \frac{q-k}{q}\frac{\bar{k}+\delta_{n,m}}{v_{m+1}^n}\right)$$

$$\mathcal{Q}(\bar{k}+\delta_{n,m+1}) = -\frac{k^2}{q\mathcal{V}_m^n}\left((\bar{k}+\alpha_{n,m})\frac{v_m^n}{q} + \frac{v_m^n}{v_{m+1}^n}\beta_{n,m}\right) + \frac{\mathcal{V}_m^{n+1}}{\mathcal{V}_m^n}\left(\gamma_{n,m}\frac{v_m^n}{q} + (\bar{k}+\delta_{n,m})\frac{v_m^n}{v_{m+1}^n}\right)$$

where

$$\mathcal{P} = \left(1 - \frac{k^2}{p^2}\right)^{-1} \qquad \mathcal{Q} = \left(1 - \frac{k^2}{q^2}\right)^{-1}.$$

The problem is reduced to checking these eight equations. From representation of $\alpha_{n,m}$, $\beta_{n,m}$, $\gamma_{n,m}$, $\delta_{n,m}$ and $\mathcal{V}_m^n$ this is done explicitly. $\qquad\square$

As mentioned, $\hat{\theta}_m^n = \pi$ is a solution of the dDSG with unstable direction. In this case the potential function on the Lax pair becomes $v_m^n = 1$, thus the solutions of (3.4a), (3.4b) takes the form

$$\phi_m^{n(-)} = (1, -k)^\perp \lambda_-^m \qquad \phi_m^{n(+)} = (1, k)^\perp b^n \lambda_+^m \qquad (3.17)$$

where

$$\lambda_\pm = a \pm \frac{ka}{q} \qquad a := \frac{q}{q-k} \qquad b := \frac{p+k}{p-k} \qquad (3.18)$$

and $k = k_d$ is a double critical point of Δ this means that satisfies the conditions

$$\Delta(k) = 2D \qquad \frac{\mathrm{d}}{\mathrm{d}k}\Delta(k) = 0$$

with

$$\Delta(k) = \left(a - \frac{ka}{q}\right)^m + b^n k\left(a + \frac{ka}{q}\right)^m \qquad |M_{n,m}| := 2b^n k\left(a^2 - \left(\frac{ka}{q}\right)^2\right)^m$$

and

$$\det M_{n,M}(\phi^-, \phi^+) = D^2 \det M_{n,0} \qquad \text{with} \quad D^2 := \prod_{m=0}^{M-1}(\lambda_- \lambda_+)^m. \qquad (3.19)$$

Inserting (3.17) into (3.12) and (3.13), and making use of the identity $\tan^{-1} X = \frac{1}{2i} \ln \left[\frac{1+iX}{1-iX} \right]$, we obtain the following analytic expression of homoclinic orbits:

$$\Theta_m^n = \pi + 4 \tan^{-1} \left(P \cos \left(m h w_1 + x \right) \operatorname{sech} \left(n h w_2 + \tau \right) \right) \tag{3.20}$$

with

$$P^2 = \frac{\sinh^2 w_2 h}{\sin^2 w_1 h} \qquad \left(1 - \tfrac{1}{4} h^2 \right) \cosh w_2 h = \left(1 + \tfrac{1}{4} h^2 \right) \cos w_1 h \qquad h = 2\pi / M \tag{3.21}$$

and $\lim_{n \to \pm\infty} \Theta_m^n = \pi$.

Remark 1. *The expression of homoclinic orbits (3.20) with the conditions (3.21) satisfies the dDSG equation. We rewrite the homoclinic solutions as follows:*

$$\Theta_m^n = 4 \tan^{-1} \left(A_m^n \right) \qquad A_m^n = P \cos \left(m h w_1 + x \right) \operatorname{sech} \left(n h w_2 + \tau \right). \tag{3.22}$$

Substitution of the solutions (3.22) into (2.2) and the identity $\tan^{-1} X = \frac{1}{2i} \ln \left[\frac{1+iX}{1-iX} \right]$ yields an equation of the form

$$\left[1 - \tfrac{1}{4} h^2 \right] \frac{1 - A_{m+1}^n A_{m-1}^n}{A_{m+1}^n + A_{m-1}^n} = \left[1 + \tfrac{1}{4} h^2 \right] \frac{1 - A_m^{n+1} A_m^{n-1}}{A_m^{n+1} + A_m^{n-1}}.$$

Using the conditions (3.21) and after some manipulations we obtain the desired identity.

Remark 2. *The continuum limit of the discrete homoclinic orbits (3.20) coincides with the homoclinic orbits of the field SG [6, 21].*

3.3. The gradient of $\Delta(k, \theta_m^n)$

The trace of the fundamental matrix of the Lax pair is an important invariant of motion for the sine-Gordon equation. The corresponding invariants $\mathcal{F}_j(\theta_m^n)$ for the dDSG system are defined as

$$\mathcal{F}_j(\theta_m^n) = \frac{1}{D} \Delta(k_j^c(\theta_m^n), \theta_m^n, \theta_{m+1}^n) \qquad v_m^n = e^{2i\theta_m^n}. \tag{3.23}$$

We shall use the invariants $\mathcal{F}_j$ to build the Mel'nikov functions, in section 4.

Lemma 1. *Let $k_j^c(\theta_m^n, \theta_{m+1}^n)$ be a simple critical point of Δ;*

$$\frac{\delta \mathcal{F}_j}{\delta \theta_m^n}(\theta_m^n, \theta_{m+1}^n) = \frac{1}{D} \frac{\delta \Delta}{\delta \theta_m^n}(k_j^c(\theta_m^n), \theta_m^n, \theta_{m+1}^n)$$

$$\frac{\delta \mathcal{F}_j}{\delta \theta_{m+1}^n}(\theta_m^n, \theta_{m+1}^n) = \frac{1}{D} \frac{\delta \Delta}{\delta \theta_{m+1}^n}(k_j^c(\theta_m^n), \theta_m^n, \theta_{m+1}^n)$$

where

$$\frac{\delta \Delta}{\delta \theta_m^n}(k, \theta_m^n) = \frac{2i}{q - k} \operatorname{trace} \left[M_{n,m+1}^{-1} \begin{pmatrix} 0 & 0 \\ \dfrac{k^2}{v_m^n} & -q \dfrac{v_{m+1}^n}{v_m^n} \end{pmatrix} M_{n,m} M_{n,M} \right]$$

$$\frac{\delta \Delta}{\delta \theta_{m+1}^n}(k, \theta_m^n) = \frac{2i}{q - k} \operatorname{trace} \left[M_{n,m+1}^{-1} \begin{pmatrix} 0 & v_{m+1}^n \\ 0 & q \dfrac{v_{m+1}^n}{v_m^n} \end{pmatrix} M_{n,m} M_{n,M} \right] \tag{3.24}$$

$v_m^n = e^{2i\theta_m^n}$ *and* $M_{n,m} = \left[\phi_m^{n(-)}, \phi_m^{n(+)} \right]$, $\phi_m^{n,(\pm)} = (\phi_m^{n(\pm,1)}, \phi_m^{n(\pm,2)})$ *are two Bloch functions of the Lax pair (3.4a), (3.4b) and* $|M_{n,m}| = \det M_{n,m}$.

Proof. Since k_j^c be a critical point, $\Delta'(k_j^c(\theta_m^n), \theta_m^n) = 0$ and

$$\frac{\delta k_j^c}{\delta \theta_m^n} = -\frac{1}{\Delta''}\frac{\delta \Delta'}{\delta \theta_m^n}$$

then k_j^c is a differential function

$$D\frac{\delta \mathcal{F}_j}{\delta \theta_m^n}(k, \theta_m^n) = \frac{\delta \Delta}{\delta \theta_m^n}\bigg|_{k=k_j^c} + \frac{\delta \Delta}{\delta k}\bigg|_{k=k_j^c}\frac{\delta k_j^c}{\delta \theta_m^n} = \frac{\delta \Delta}{\delta \theta_m^n}\bigg|_{k=k_j^c}.$$

Let $M_{n,m}$ be the fundamental matrix of the Lax pair, for the special potential function $v_m^n = \mathrm{e}^{2\mathrm{i}\theta_m^n}$. Variation of the v_m^n leads to the variational equation for the variation of $M_{n,m}$ at fixed k:

$$M_{n,m+1} = V_{n,m}M_{n,m}$$

$$\delta M_{n,m+1} = V_{n,m}\delta M_{n,m} + \delta V_{n,m}M_{n,m}$$

$$\delta V_{n,m} = \begin{pmatrix} 0 & \dfrac{1}{q-k}\delta v_{m+1}^n \\[2ex] \dfrac{k^2}{q-k}\delta\left(\dfrac{1}{v_m^n}\right) & \dfrac{q}{q-k}\delta\left(\dfrac{v_{m+1}^n}{v_m^n}\right) \end{pmatrix}.$$

Let $\delta M_{n,m} = M_{n,m}A_{n,m}$, where $A_{n,m}$ is the 2×2 matrix to be determined.

$$\delta M_{n,m+1} = M_{n,m+1}A_{n,m+1} = V_{n,m}M_{n,m}A_{n,m} + \delta V_{n,m}M_{n,m}$$

$$= M_{n,m+1}A_{n,m} + \delta V_{n,m}M_{n,m}$$

thus,

$$A_{n,m+1} - A_{n,m} = M_{n,m+1}^{-1}\delta V_{n,m}M_{n,m}$$

$$A_{n,0} = 0. \tag{3.25}$$

Solving the system (3.25) we have,

$$\delta M_{n,m} = M_{n,M}\left[\sum_{J=1}^{M-1} M_{n,J}^{-1}\delta V_{n,J-1}M_{n,J-1}\right] \qquad \delta M_{n,0} = 0.$$

Then,

$$\delta \Delta(k, \theta_m^n) = \mathrm{trace}\left\{M_{n,M}\left[\sum_{J=1}^{M-1} M_{n,J}^{-1}\delta V_{n,J-1}M_{n,J-1}\right]\right\}$$

thus, we obtain equations (3.24). Substitution the representation of $M_{n,m}$ into (3.24) entails

$$\frac{\delta \Delta}{\delta \theta_m^n}(k, \theta_m^n) = \frac{2\mathrm{i}}{q-k}\frac{1}{|M_{n,m+1}|}\left\{-\frac{k^2}{v_m^n}\left[\phi_{m+1}^{n(+1)}\left(\phi_m^{n(-1)}\phi_M^{n(-1)} + \phi_m^{n(+1)}\phi_M^{n(-2)}\right)\right.\right.$$

$$+\phi_{m+1}^{n(-2)}\left(\phi_m^{n(-1)}\phi_M^{n(+1)} + \phi_m^{n(+1)}\phi_M^{n(+2)}\right)\big]$$

$$+q\frac{v_{m+1}^n}{v_m^n}\left[\phi_{m+1}^{n(-1)}\left(\phi_m^{n(-1)}\phi_M^{n(+1)} + \phi_m^{n(+2)}\phi_M^{n(+2)}\right)\right.$$

$$\left.\left.-\phi_{m+1}^{n(+1)}\left(\phi_m^{n(-2)}\phi_M^{n(-1)} + \phi_m^{n(+2)}\phi_M^{n(-2)}\right)\right]\right\} \tag{3.26a}$$

$$\frac{\delta \Delta}{\delta \theta_{m+1}^n}(k, \theta_m^n) = \frac{2\mathrm{i}}{q-k}\frac{1}{|M_{n,m+1}|}\left\{v_{m+1}^n\left[\phi_m^{n(+2)}\left(\phi_m^{n(-2)}\phi_M^{n(-1)} + \phi_m^{n(+2)}\phi_M^{n(-2)}\right)\right.\right.$$

$$\left.\left.-\phi_m^{n(-2)}\left(\phi_m^{n(-2)}\phi_M^{n(+1)} + \phi_m^{n(+2)}\phi_M^{n(+2)}\right)\right]\right]$$

$$+q\frac{v_{m+1}^n}{v_m^n}\big[\phi_m^{n(-1)}\big(\phi_m^{n(-2)}\phi_M^{n(+1)}+\phi_m^{n(+2)}\phi_M^{n(+2)}\big)$$

$$-\phi_m^{n(+1)}\big(\phi_m^{n(-2)}\phi_M^{n(-1)}+\phi_m^{n(+2)}\phi_M^{n(-2)}\big)\big]\Big\}.\qquad (3.26b)$$

Substitute (3.17) and (3.20) into (3.26a) and (3.26b), we obtain the expressions of $\delta\Delta/\delta\theta_m^n$ evaluated on the homoclinic orbits. This completes the proof of the lemma.   □

4. Homoclinic orbits for the perturbed dDSG equation

Next, we will establish the persistence of homoclinic orbits (3.20) for the perturbed dDSG equation (2.8). First, we show that the solution $\hat{\theta}_m^n = \pi$ is a singular fixed point of the integrable dDSG equation and second, we prove that, for sufficiently small $\varepsilon > 0$, there exists a solution homoclinic to $\hat{\theta}_m^n(\varepsilon)$.

4.1. Linearized analysis

In this subsection, we discuss the linear stability analysis of the fixed solutions of dDSG (2.8). We note that if θ_m^n solves equation (2.3) then $(-1)^{n+m}\theta_m^n$ is also a solution. Assume that the small perturbation $\hat{\theta}_m^n(\varepsilon)$ are exponentially fast growing, as a consequence of the linearly unstable modes of the continuum SG equation [21]

$$\theta_m^n = \hat{\theta}_m^n(\varepsilon) + \eta\varphi_m^n \qquad 0 < \eta \ll 1 \qquad (4.1a)$$

with

$$\varphi_m^n = \Omega_k^n \exp\big[\mathrm{i}\mathcal{A}_k m\big] \qquad \mathcal{A}_k = \frac{2\pi k}{M} \qquad (4.1b)$$

where $\hat{\theta}_m^n(\varepsilon) = \pi + \mathcal{O}(\varepsilon)$ is a solution of (2.8), $M = 2p$ the total number of grid points in the space interval $(0, 1)$ and $k = -M/2, \ldots, M/2 - 1$.

Linearizing equation (2.8) we obtain

$$\frac{\varphi_m^{n+1} - 2\varphi_m^n + \varphi_m^{n-1}}{h^2} - \frac{\varphi_{m+1}^n - 2\varphi_m^n + \varphi_{m-1}^n}{h^2} - \tfrac{1}{4}\big(\varphi_m^{n+1} + \varphi_m^{n-1} + \varphi_{m+1}^n + \varphi_{m-1}^n\big)$$

$$= \varepsilon\bigg\{b\,\varphi_m^n + c\,\frac{\varphi_{m+1}^n - 2\varphi_m^n + \varphi_{m-1}^n}{h^2}\bigg\}. \qquad (4.2)$$

After straightforward calculations, we take the following equation for eigenvalues Ω_k:

$$\Omega_k^2 - 2\bigg(\frac{\cos\mathcal{A}_k}{a} + \varepsilon\big(\tfrac{1}{2}bh^2 - c\sin^2\tfrac{1}{2}\mathcal{A}_k\big)\bigg)\Omega_k + 1 = 0 \qquad (4.3)$$

with

$$a = \big(1 - \tfrac{1}{4}h^2\big)\big(1 + \tfrac{1}{4}h^2\big)^{-1} \qquad h = 2\pi/M \qquad (4.4)$$

and

$$\Omega_k = \bigg(\frac{\cos\mathcal{A}_k}{a} + \varepsilon\big(\tfrac{1}{2}bh^2 - c\sin^2\tfrac{1}{2}\mathcal{A}_k\big)\bigg) \pm \sqrt{\bigg|\bigg(\frac{\cos\mathcal{A}_k}{a} + \varepsilon\big(\tfrac{1}{2}bh^2 - c\sin^2\tfrac{1}{2}\mathcal{A}_k\big)\bigg)^2 - 1\bigg|}.$$

$$(4.5)$$

The stability condition $|\Omega_k| \leqslant 1$ becomes

$$\bigg(\frac{\cos\mathcal{A}_k}{a} + \varepsilon\big(\tfrac{1}{2}bh^2 - c\sin^2\tfrac{1}{2}\mathcal{A}_k\big)\bigg)^2 \leqslant 1. \qquad (4.6)$$

It follows after some manipulations that the dispersion relation at $\mathcal{A}_k = 2\pi k/M$ may be written as follows:

$$\Omega_k = \frac{\cos \mathcal{A}_k}{a} + \varepsilon\left(\tfrac{1}{2}bh^2 - c\sin^2 \tfrac{1}{2}\mathcal{A}_k\right) \pm 2\mathrm{i}\cos^2 \tfrac{1}{2}\mathcal{A}_k\sqrt{E_k F_k} \tag{4.7}$$

for $k = -M/2, \ldots, M/2 - 1, k \neq 0$, and

$$E_k := \tfrac{1}{4}h^2 - \tan^2 \tfrac{1}{2}\mathcal{A}_k + \varepsilon\frac{1 - \tfrac{1}{4}h^2}{2\cos^2 \tfrac{1}{2}\mathcal{A}_k}\left(\tfrac{1}{2}bh^2 - c\sin^2 \tfrac{1}{2}\mathcal{A}_k\right) + \mathcal{O}(\varepsilon^2)$$

$$F_k := 1 - \tfrac{1}{4}h^2 \tan^2 \tfrac{1}{2}\mathcal{A}_k + \varepsilon\frac{1 - \tfrac{1}{4}h^2}{2\cos^2 \tfrac{1}{2}\mathcal{A}_k}\left(\tfrac{1}{2}bh^2 - c\sin^2 \tfrac{1}{2}\mathcal{A}_k\right) + \mathcal{O}(\varepsilon^2). \tag{4.8}$$

The stability condition (4.6) is violated by the wavenumber $\mathcal{A}_k \simeq 0$ for $k = 0$. The unstable wavenumbers satisfy

$$\left(\frac{\cos \mathcal{A}_k}{a} + \varepsilon\left(\tfrac{1}{2}bh^2 - c\sin^2 \tfrac{1}{2}\mathcal{A}_k\right) - 1\right)\left(\frac{\cos \mathcal{A}_k}{a} + \varepsilon\left(\tfrac{1}{2}bh^2 - c\sin^2 \tfrac{1}{2}\mathcal{A}_k\right) + 1\right) > 0 \tag{4.9a}$$

which implies the following:

$$\tan^2 \tfrac{1}{2}\mathcal{A}_k < \tfrac{1}{4}h^2\left(1 + \varepsilon(1 - \tfrac{1}{4}h^2)\left(\frac{b}{\cos \mathcal{A}_k/2} - \frac{c}{2}\right)\right) + \mathcal{O}(\varepsilon^2). \tag{4.9b}$$

For $k = 0$, the perturbed fixed point $\hat{\theta}_m^n(\varepsilon)$ loses stability and is turned into an *unstable hyperbolic fixed point* with

$$\Omega_0 = \frac{1}{a} + \varepsilon\tfrac{1}{2}bh^2 \pm h\sqrt{\left(1 + \varepsilon(1 - \tfrac{1}{4}h^2)b\right)\left(1 + \varepsilon(1 - \tfrac{1}{4}h^2)\tfrac{1}{4}bh^2\right)}. \tag{4.10}$$

4.2. Mel'nikov method

With the geometric structures constructed in the previous sections, we start to construct orbits for the perturbed dDSG (2.8) homoclinic to the saddle point $\hat{\theta}_m^n(\varepsilon)$. The construction is composed of two steps. We start with an initial point in the unstable manifold of $\hat{\theta}_m^n(\varepsilon)$, and first we show that the forward orbit enters the centre-stable manifold of the neighbourhood $\mathcal{M}_\varepsilon$, if the parameters b, c are properly chosen. Second, we prove that the perturbed orbits approach the fixed point $\hat{\theta}_m^n(\varepsilon)$ in forward time $n \to +\infty$. The Mel'nikov method is the main tool.

We write the perturbed dDSG equation (2.8) in the form

$$\theta_m^{n+1} = \boldsymbol{F}(\theta_m^{n-1}, \theta_{m\pm1}^n) + \varepsilon\boldsymbol{G}(\theta_m^n, \theta_{m\pm1}^n) \tag{4.11}$$

where $\boldsymbol{G}$ denotes the perturbation

$$\boldsymbol{G} := b\,\theta_m^n + c\,\frac{\theta_{m+1}^n - 2\theta_m^n + \theta_{m-1}^n}{h^2}.$$

Before we discuss the persistent homoclinic orbits, we make some observations related to the existence of an invariant subspace for the dDSG. In particular, the dDSG equation admits an invariant subspace Π. Indeed, if we consider the constraint

$$\theta_{m+1}^n = \theta_{m-1}^n = \theta_m^n \qquad m = 0, 1, \ldots, M-1 \quad M = 2p. \tag{4.12}$$

On the plane Π the perturbed system has a saddle fixed point $\hat{\theta}_m^n(\varepsilon)$ and in the full phase space linear stability analysis shows that $\hat{\theta}_m^n(\varepsilon)$ is a saddle point and by the invariant manifold

theory $W^{\mathrm{u}}(\hat{\theta}_m^n(\varepsilon))$ exists and is two dimensional (two unstable directions in and off the plane Π). $W^{\mathrm{s}}(\hat{\theta}_m^n(\varepsilon))$ also exists and has codimension two and $W_\varepsilon^{\mathrm{cs}}$ has codimension one. The intersection $W^{\mathrm{u}}(\hat{\theta}_m^n(\varepsilon)) \cap W_\varepsilon^{\mathrm{cs}}$ will be one dimensional.

Thus, the unstable, stable and centre manifolds of the map $\boldsymbol{F}$ (cf (2.5)) associated with $\hat{\theta}_m^n$:

$$W^{\mathrm{u}} = \left\{\theta_m^n \in \mathcal{L} : \lim_{n \to -\infty} \boldsymbol{F}^n(\theta_m^n) = \hat{\theta}_m^n\right\}$$
$$W^{\mathrm{s}} = \left\{\theta_m^n \in \mathcal{L} : \lim_{n \to +\infty} \boldsymbol{F}^n(\theta_m^n) = \hat{\theta}_m^n\right\} \tag{4.13}$$

with $W^{\mathrm{s}}(\hat{\theta}_m^n(\varepsilon)) \subset W_\varepsilon^{\mathrm{cs}}$ and $\mathcal{M}_\varepsilon$ being a neighbourhood such that the whole plane Π lies inside $\mathcal{M}_\varepsilon$.

For the perturbed system, we have that the local stable and unstable manifolds $W_{\varepsilon,\mathrm{loc}}^{\mathrm{s,u}}$ of the perturbed point $\hat{\theta}_m^n(\varepsilon)$ are 'ε-close' to those of the unperturbed point $\hat{\theta}_m^n = \pi$.

Consider the saddle fixed point $\hat{\theta}_m^n(\varepsilon)$ on the invariant plane Π and we seek an orbit, not on Π, which is homoclinic to $\hat{\theta}_m^n(\varepsilon)$. We present a geometrical mean to establish the persistence of homoclinic orbits for the near-integrable dDSG equation. Our method based on the Mel'nikov measurement allows us to answer the question: is there any intersection between $W_{\mathrm{loc}}^{\mathrm{u}}(\hat{\theta}_m^n(\varepsilon))$ and $W^{\mathrm{s}}(\mathcal{M}_\varepsilon) \subset W_\varepsilon^{\mathrm{cs}}$?

The homoclinic orbits of the dDSG equation approach the subspace

$$\Pi = \left\{\theta_m^n \in \mathcal{L} : \theta_{m+1}^n = \theta_{m-1}^n = \theta_m^n, \ m = 0, 1, \ldots, M-1\right\}$$

as $n \to +\infty$. Our analysis of the near-integrable dDSG equation begins with two observations: *First*, when $\varepsilon = 0$, the lattice SG equation is a completely integrable system. *Second*, the subspace Π is an invariant space for perturbed dDSG. In each of these two cases ($\varepsilon = 0$ or $\theta_m^n \in \Pi$) the behaviour of solution θ_m^n (as $n \in \mathbb{Z}$) can be described completely. In the first case, this description is described in section 3, through the Bäcklund transformation of the solution π and the Lax pair, in the second case it is accomplished through a phase plane analysis of the reduced map (discrete pendulum). In the jargon of the theory of dynamical systems our method will be a form of a 'local–global' analysis, where at times the term 'local' will mean close to the subspace Π of the discrete pendulum, and at other times 'local' will mean close to the integrable solutions, homoclinic with respect to the index n. In any event, throughout the global arguments (the Mel'nikov method), control is achieved either because of proximity to (a) the subspace Π or (b) $\varepsilon = 0$.

The essence of the Mel'nikov method is as follows. Firstly, we establish the intersection of $W_\varepsilon^{\mathrm{cs}}$ and $W_{\varepsilon,\mathrm{loc}}^{\mathrm{u}}(\hat{\theta}_m^n(\varepsilon))$ of the perturbed dDSG equation, so that for external parameters in a fixed open set, there are orbits which tend to $\hat{\theta}_m^n(\varepsilon)$ when $n \to -\infty$ and approach Π when $n \to +\infty$. Secondly, we prove that these orbits also tend to $\hat{\theta}_m^n(\varepsilon)$ in forward 'time' ($n \to +\infty$).

Let h_m^n be the homoclinic orbits of the dDSG and $h_m^{n,\mathrm{u}}(\varepsilon)$, $h_m^{n,\mathrm{s}}(\varepsilon)$ orbits lie on $W_{\varepsilon,\mathrm{loc}}^{\mathrm{u}}(\hat{\theta}_m^n(\varepsilon))$ and $W_{\varepsilon,\mathrm{loc}}^{\mathrm{s}}(\hat{\theta}_m^n(\varepsilon))$, respectively, ε-close to h_m^n. We give the following parametrization for these orbits:

$$\begin{aligned}
h_m^{n,\mathrm{u}}(\varepsilon) \quad &n \in \mathbb{Z}_- & h_m^{0,\mathrm{u}}(\varepsilon) &= q_h^{\mathrm{u},\varepsilon} \in W_{\varepsilon,\mathrm{loc}}^{\mathrm{u}} \\
h_m^n(0) \quad &n \in \mathbb{Z} & h_m^0(0) &= q_h \\
h_m^{n,\mathrm{s}}(\varepsilon) \quad &n \in \mathbb{Z}_+ & h_m^{0,\mathrm{s}}(\varepsilon) &= q_h^{\mathrm{s},\varepsilon} \in W_{\varepsilon,\mathrm{loc}}^{\mathrm{s}}.
\end{aligned} \tag{4.14}$$

We define the following distance between $q_h^{u,\varepsilon}$, $q_h^{s,\varepsilon}$ along the tangent vector v_0 at the point h_m^0:

$$d = \text{distance}(q_h^{u,\varepsilon}, q_h^{s,\varepsilon}) = \sum_{m=0}^{M-1} v_0(h_m^0) \cdot \left(h_{m+1}^{0,u}(\varepsilon) - h_{m+1}^{0,s}(\varepsilon)\right)$$

$$= \langle v_0, h_m^{0,u}(\varepsilon) - h_m^{0,s}(\varepsilon) \rangle = \mathcal{D}^u(\varepsilon) - \mathcal{D}^s(\varepsilon) \tag{4.15}$$

where the tangent vector v_0 is defined by the gradient of the function Δ (cf (3.24)). We define

$$\mathcal{D}^s(\varepsilon) = \langle v_0(h_m^0), h_m^{0,s}(\varepsilon) - h_m^{0,u}(\varepsilon) \rangle$$

$$\mathcal{D}^u(\varepsilon) = \langle v_0(h_m^0), h_m^{0,u}(\varepsilon) - h_m^{0,u}(\varepsilon) \rangle. \tag{4.16}$$

The orbits h_m^n, $h_m^{n,u}$, $h_m^{n,s}$ fulfil the equations

$$h_m^{n+1} = \boldsymbol{F}(h_m^{n-1}, h_{m+1}^n, h_{m-1}^n)$$

$$h_m^{n+1,u}(\varepsilon) = \boldsymbol{F}(h_m^{n,u}(\varepsilon)) + \varepsilon \boldsymbol{G}(h_m^{n,u}(\varepsilon)) \tag{4.17}$$

$$h_m^{n+1,s}(\varepsilon) = \boldsymbol{F}(h_m^{n,s}(\varepsilon)) + \varepsilon \boldsymbol{G}(h_m^{n,s}(\varepsilon))$$

where $\boldsymbol{F}$ and $\boldsymbol{G}$ are defined explicitly in (4.11). We study the motion on the unstable manifold. From (4.17) we have

$$y_m^{n+1} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} h_m^{n+1,u}(\varepsilon)\Big|_{\varepsilon=0} = D\boldsymbol{F}(h_m^{n,u}) y_m^{n,u} + \boldsymbol{G}(h_m^{n,u}) \tag{4.18}$$

then

$$\langle v_m^{n+1}, y_m^{n+1} \rangle = \langle v_m^{n+1}, D\boldsymbol{F}(h_m^n) y_m^n \rangle + \langle v_m^{n+1}, \boldsymbol{G}(h_m^{n,u}) \rangle.$$

Using the integrability of the unperturbed system $\boldsymbol{F}$ and the mathematical induction, we obtain

$$\langle v_m^0, y_m^{0,u} \rangle = \sum_{n=-\infty}^{0} \sum_{m=0}^{M-1} v_m^n \, \boldsymbol{G}(h_{m+1}^{n-1}) \tag{4.19a}$$

and similarly for the stable manifold

$$\langle v_m^0, y_m^{0,s} \rangle = \sum_{n=1}^{\infty} \sum_{m=0}^{M-1} v_m^n \, \boldsymbol{G}(h_{m+1}^{n-1}). \tag{4.19b}$$

Then

$$d = \varepsilon(\mathcal{D}^u(0) - \mathcal{D}^s(0)) + \mathcal{O}(\varepsilon^2) = \varepsilon M_{\mathcal{F}} + \mathcal{O}(\varepsilon^2) \tag{4.20}$$

where

$$M_{\mathcal{F}} := \sum_{n\in\mathbb{Z}} \sum_{m=0}^{M-1} v_m^n(h_m^n) \cdot \boldsymbol{G}(h_m^{n-1}) \tag{4.21}$$

with v_m^n defined in (3.26a), (3.26b), and evaluated on the homoclinic orbit h_m^n.

The Mel'nikov function (4.21) becomes

$$M_{\mathcal{F}}(x, \tau; h, M) = bS_1 + cS_2 \tag{4.22a}$$

with

$$S_1(x, \tau; h, M) = \sum_{n\in\mathbb{Z}} \sum_{m=0}^{M-1} \frac{\delta\mathcal{F}}{\delta\theta_m^n}(h_m^n) \cdot h_m^{n-1}$$

$$S_2(x, \tau; h, M) = \sum_{n\in\mathbb{Z}} \sum_{m=0}^{M-1} \frac{\delta\mathcal{F}}{\delta\theta_m^n}(h_m^n) \cdot \frac{h_{m+1}^{n-1} - 2h_m^{n-1} + h_{m-1}^{n-1}}{h^2}. \tag{4.22b}$$

Setting $M_{\mathcal{F}} = 0$ we obtain an algebraic equation which describe a two-dimensional surface on the external parameter space (b, c):

$$bS_1 + cS_2 = 0. \tag{4.23}$$

We make the following observation, the only singularities of the homoclinic orbits $\Theta_m^n \, (= h_m^n)$ are simple poles at any point $\tau_\theta \in i\pi/2 + i\pi\mathbb{Z}$ and therefore Θ_m^n is analytic at $\tau_\theta + hk$ for $k \in \mathbb{Z} \setminus \{0\}$.

It is important to bear in mind that our model admits an invariant subspace Π and the persistent homoclinic orbit will leave the saddle point $\hat{\theta}_m^n(\varepsilon)$ near Π, rapidly fly away and return along a global orbit which is close to one of the integrable homoclinic orbits and must approach the saddle point. In order to investigate this fact, we focus our attention to the dynamics of the perturbed discrete pendulum in Π. The dynamics of dDSG on Π is governed by the following standard map (so-called discrete pendulum):

$$\theta^{n+1} = -\theta^{n-1} + 2\theta^n - 4\operatorname{Arg}\left(1 + \tfrac{1}{4}h^2 e^{i\theta^n}\right) + \varepsilon b\theta^n \tag{4.24}$$

or

$$f_\Pi(x, y) = \left(y, -x + 2y - 4\operatorname{Arg}\left(1 + \tfrac{1}{4}h^2 e^{iy}\right)\right) + \varepsilon(0, by) := f_0 + \varepsilon f_1.$$

We observe that the mapping f_0 is an area preserving with the following generating function:

$$\mathcal{S}_0(x, X) = -xX + X^2 + \tilde{F}(X + \sigma) + \tilde{F}(\sigma - X) \tag{4.25}$$

with

$$\tilde{F}(x) = 2i \int_{-\infty}^{x} \ln(1 + e^{i\xi})\,d\xi \qquad e^{i\sigma} = h^2/4.$$

The solution $\hat{\theta}^n = \pi$ is a saddle fixed point of the unperturbed map f_0, since the residues $\mathcal{R}$ of the Jacobian matrix $Df_0(\pi)$ passes through zero:

$$\mathcal{R} = \tfrac{1}{4}\left(2 - \operatorname{trace} Df_0(\pi)\right) = \frac{h^2}{h^2 - 4} < 0 \qquad 0 < h < 1.$$

Another feature of the discrete pendulum is that it admits a one-parameter ($\tau \in \mathbb{R}$) family of homoclinic orbits to the saddle point $\hat{\theta}^n = \pi$:

$$\begin{aligned}
\theta^n(\tau) &= \pi + 4\tan^{-1}\left(\sinh w \cos x \operatorname{sech}(nwh + \tau)\right) \\
\theta^{n+1}(\tau) &= \pi + 4\tan^{-1}\left(\sinh w \cos x \operatorname{sech}((n+1)wh + \tau)\right)
\end{aligned} \tag{4.26}$$

with

$$w = \sinh^{-1}\mu \qquad \text{and} \qquad \mu := h^2\left(\tfrac{1}{4}h^2 - 1\right)^{-1} \tag{4.27}$$

One can verify after some manipulations that the expression (4.26) satisfies the map f_0 and

$$\lim_{n \to \pm\infty} \theta^n = \pi$$

with π being a saddle point of the map f_0. A third aspect of the discrete pendulum system is directly related to the integrability. The unperturbed map (4.24) has the following integral of motion:

$$H(\theta^n, \theta^{n-1}) = 2(\cos\theta^n + \cos\theta^{n-1}) + \frac{h^2}{4}\cos(\theta^n + \theta^{n-1}) + \frac{4}{h^2}\cos(\theta^n - \theta^{n-1}). \tag{4.28}$$

Moreover, the dynamics of (4.24) with ε small is ε-close to the dynamics of the discrete pendulum. These properties of the map (4.24) are important in establishing the persistence of homoclinic orbits for the dDSG equation.

We have proved that there is a generic intersection between $W^{\mathrm{u}}(\hat\theta^n_m(\varepsilon))$ and $W^{\mathrm{s}}(\mathcal{M}_\varepsilon) \subset \mathcal{M}_\varepsilon$, so that for external parameters (b, c) in a fixed open set there are orbits $h^n_m(\varepsilon)$ which tend to $\hat\theta^n_m(\varepsilon)$ as $n \to -\infty$ and approach $\mathcal{M}_\varepsilon$ in forward time $n \to +\infty$. Now, we prove that these orbits in forward time also tend to $\hat\theta^n_m(\varepsilon)$.

Let

$$W^{\mathrm{s}}(\hat\theta^n_m(\varepsilon))|_\Pi = \Pi \cap W^{\mathrm{s}}(\hat\theta^n_m(\varepsilon))$$
$$W^{\mathrm{s,u}}(\hat\theta^n_m(\varepsilon))|_{\mathcal{M}_\varepsilon} = \mathcal{M}_\varepsilon \cap W^{\mathrm{s,u}}(\hat\theta^n_m(\varepsilon))$$

and from the linearized analysis we have $\dim W^{\mathrm{s,u}}(\hat\theta^n_m(\varepsilon))|_\Pi = 1$, $\operatorname{codim} W^{\mathrm{s}}(\hat\theta^n_m(\varepsilon))|_{\mathcal{M}_\varepsilon} = 1$ in $\mathcal{M}_\varepsilon$. Let $q^{\mathrm{s},\varepsilon}$ denote the intersection point of the orbit $h^n_m(\varepsilon)$ with the boundary of the set U (let U be a neighbourhood of the phase space of (4.24), then $q^{\mathrm{s},\varepsilon} \in \partial U \cap W^{\mathrm{s}}(\mathcal{M}_\varepsilon)$, with projection point $\pi(q^{\mathrm{s},\varepsilon}) \in \mathcal{M}_\varepsilon$. From the invariant manifolds theory, the points $q^{\mathrm{s},\varepsilon}$ and $\pi(q^{\mathrm{s},\varepsilon})$ will approach the same point in forward time. Also, we have

$$\operatorname{distance}\big(q^{\mathrm{s},\varepsilon}, \pi(q^{\mathrm{s},\varepsilon})\big) = \mathcal{O}(\varepsilon).$$

We measure the distance between the projection point $\pi(q^{\mathrm{s},\varepsilon})$ and the point $q^{\mathrm{u},\varepsilon}_\Pi \in W^{\mathrm{u}}(\hat\theta^n_m(\varepsilon))|_\Pi$

$$d := \operatorname{distance}(\pi(q^{\mathrm{s},\varepsilon}), q^{\mathrm{u},\varepsilon}_\Pi) = \varepsilon M_H(\tau, w) + \mathcal{O}(\varepsilon^2). \tag{4.29}$$

The Mel'nikov sum M_H is built with the constant of motion H of the discrete pendulum (cf (4.28)):

$$M_H(\tau; h) = \sum_{n\in\mathbb{Z}} g(\tau + nwh) \tag{4.30}$$

where

$$g := b\frac{\partial H}{\partial \theta^n}\big(\theta(\tau + nwh)\big)\,\theta(\tau + nhw)$$
$$\frac{\partial H}{\partial \theta^n} = -2\sin\theta^n - \tfrac{1}{4}h^2\sin(\theta^n + \theta^{n-1}) - \frac{4}{h^2}\sin(\theta^n - \theta^{n-1}) \tag{4.31}$$

and $\theta^n(\tau)$ is given in (4.26). The above sum reduces to compute the residues of the function g. In particular, the function that plays a significant role in the computation of the infinite sum (4.30) is a complex function χ satisfying the following properties: χ is meromorphic in $\mathbb{C}$, is $T\mathrm{i}$-periodic and w-periodic and the set of poles of χ is $w\mathbb{Z} + T\mathrm{i}\mathbb{Z}$, all of them being simple and of residue 1. Also, the function g appears in formula (4.30) verifying: (a) it is analytic in $\mathbb{R}$ and has only isolated singularity in $\mathbb{C}$; (b) it is $T\mathrm{i}$-periodic for $T > 0$; and (c) it fulfils $|g(\tau)| \leqslant A\exp[-c|\operatorname{Re}\tau|]$ when $|\operatorname{Re}\tau| \to \infty$ for some constants $A, c > 0$. Then, the Mel'nikov function M_H is analytic in $\mathbb{R}$, only has isolated singularities in $\mathbb{C}$ and is doubly periodic with periods $h, T\mathrm{i}$. Moreover, $M_H(\tau)$ can be expressed by the following sum:

$$M_H(\tau; h) = -\sum_{z\in\mathcal{J}} \operatorname{res}\big(\chi_T g; z\big) \tag{4.32}$$

where

$$\mathcal{J} := \big\{z \in \mathbb{C} : 0 < \operatorname{Im} z < T\big\}.$$

The two conditions of the Melnikov function $M = (M_{\mathcal{F}}, M_H)$

$$M_{\mathcal{F}} = 0 \qquad \text{and} \qquad M_H = 0 \tag{4.33}$$

will select isolated values of two variables x, τ (x, τ parametrize the homoclinic orbits). This will certainly be the case if, in addition to satisfying (4.33), x and τ also satisfy the transversality condition:

$$\det D_{(x,\tau)}M \neq 0$$

where D is the Jacobian of the Mel'nikov function. If the condition (4.33) is augmented by this transversality condition then the homoclinic orbits corresponding to these selected isolated values do, in fact, persist for $0 < \varepsilon \ll 1$, w_1, w_2 given by (4.27) and $b, c \in \mathbb{R}$. Thus there exists an exactly transversal intersection point and the perturbed orbits $h_m^n(\varepsilon)$ tend to $\hat{\theta}_m^n(\varepsilon)$ as $n \to +\infty$. We can now state the result.

Proposition 2. *Consider the near-integrable dDSG equation (2.8) with periodic boundary conditions (2.6). Let M be an even fixed positive integer and $h := 2\pi/M < 1$ be a constant real number. Then, there exists $\varepsilon_0 > 0$ such that, for any fixed parameters $\{w_1, w_2, b, c, \varepsilon\}$ there are isolated values x_0, τ_0 of two variables x, τ at the Melnikov functions $M_{\mathcal{F}}, M_H$ satisfy the following conditions:*

$$M_{\mathcal{F}} = 0 \qquad and \qquad M_H = 0$$

$$\det D_{(x,\tau)}M \neq 0.$$

The system (2.8) admits a homoclinic orbit $\theta_m^n(\varepsilon)$ that is doubly asymptotic to the m-independent solution θ^n originated from $\hat{\theta}_m^n(\varepsilon) \in \Pi$.

5. Conclusions

We have developed a geometric method of studying the behaviour of homoclinic solutions of the double discrete sine-Gordon equation with periodic boundary conditions under small dissipative perturbations. We choose a class of dDSG equation to illustrate between the methods of PΔEs and those of dynamical systems theory (symplectic mappings) by which these properties can be understood mathematically. Specifically, for the study of the global behaviour for soliton PΔEs, we advocate implementary intuition from the theory of dynamical systems with methods natural from the PΔEs. We have found an analytic expression of homoclinic orbits for dDSG under periodic boundary conditions through the Bäcklund transformation and the Lax pair formalism. We have shown that these orbits persist under dissipative perturbations through the Mel'nikov criteria for discrete dynamical systems. Moreover, our results should be interested in the construction of a new discrete surface and how these new geometric objects can be obtained through numerical simulations and a comparison with the well known discrete surfaces will be important. In future works, we will study the geometric principles and the homoclinic structure in another important class of lattice equations obtained by extending the dDSG equation to higher-order partial difference systems.

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